S.-T. Yau College Student Mathematics Contests 2023

Analysis and Differential Equations

All-around

Please solve one of the following two problems.

1. Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a C_c^∞ function. We assume $\phi \geq 0$ and $\int_{\mathbb{R}^3} \phi(x) = 1$. Consider convolution operator $Tf := f * \phi$, i.e.

$$(Tf)(x) := \int_{\mathbb{R}^3} f(x-y)\phi(y)dy. \tag{0.1}$$

Is T a bounded operator from $L^3(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$? Prove your conclusion.

2. Given discrete points $\{a_i\}_{i\geqslant 1}$ on $\mathbb{D} = \{|z| < 1\}$, construct a holomorphic function f on $\mathbb{D}$ such that the zero set of f is exactly $\{a_i\}_{i\geqslant 1}$.

Hint: there are no accumulation points in the interior.